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(September 12, 2001)

The classical evolution and the quantum generation processes of the scalar- and tensor-type cosmological perturbations in the context of a broad class of generalized gravity theories are presented in unified forms. The exact forms of final spectra of the two types of structures generated during a generalized slow-roll inflation are derived. Results in generalized gravity are characterized by two additional parameters which are the coupling between gravity and field $f(\phi, R)$, and the nonminimal coupling in the kinetic part of the field $\omega(\phi)$. Our general results include widely studied gravity theories and inflation models as special cases, and show how the well known consistency relation and spectra in ordinary Einstein gravity inflation models are affected by the generalized nature of the gravity theories.

I. INTRODUCTION

Lifshitz instability theory [1], the relativistic linear perturbation theory of an expanding Friedmann world model, first presented in 1946 has been studied in the literature over more than a half century [2–5]. The observed cosmological structures in the large-scale and in the early universe are generally believed to behave as small deviations from the homogeneous and isotropic background world model. Under such a situation the relativisitic cosmological perturbation analysis becomes manageable due to the assumed linearity of the structures. Recent observational advances of the CMBR anisotropies conform/reinforce the validity of the two basic assumptions used in most of the cosmological structure formation theories: the homogeous and isotropic Friedmann world model, and the linearity of the imposed structures.

However, the observational evidences do not necessarily constrain the underlying gravity theory, especially during the seed generating stage in the very early univese, to be Einstein one. Generalized forms of gravity appear ubiquitously in any reasonable attempts to understand the quantum aspects of the gravity theory, and also naturally appear in the low energy limits of diverse attempts to unify gravity with other fundamental forces, like the Kaluza-Klein, the supergravity, the string/M-theory programs. Modifying terms appear naturally in the quantization processes of the gravity theory in a way toward the quantum gravity. Thus, there arises a growing chance that the early stages of the universe were governed by the gravity more general than Einstein one.

Reflecting such possibilities, there have been many studies of the world models as well as the perturbations based on variety of generalized gravity theories [6–8]; for our study see [9–12]. In this paper we will present the classical evolution and quantum generation processes, and the consequent inflationary spectra in unified forms which include (1) the scalar- and the tensor-type structures, and (2) the fluid and the field in Einstein gravity, and the field in a class of generalized gravity theories.

We set $c \equiv 1$.

II. GRAVITY AND WORLD MODEL

We consider gravity theories with the following action

$$S = \int d^4 x \sqrt{-g} \Big[\frac{1}{2} f(\phi, R) - \frac{1}{2} \omega(\phi) \phi^{;a} \phi_{,a} - V(\phi) + L_m \Big], \quad (1)$$

where $f(\phi, R)$ is a general algebraic function of the scalar field ϕ and the scalar curvature R; $\omega(\phi)$ and $V(\phi)$ are general algebraic functions of ϕ. L_m is the matter Lagrangian with the hydrodynamic energy-momentum tensor T_{ab} defined as $\delta(\sqrt{-g} L_m) \equiv \frac{1}{2} \sqrt{-g} T^{ab} \delta g_{ab}$. Our generalized gravity includes as subset [10]: $f(R)$ gravity which includes R^2 gravity, the scalar-tensor theory which includes the Jordan-Brans-Dicke theory [13], the non-minimally coupled scalar field, the induced gravity [14], the low-energy effective action of string theory [15], etc. It does not, however, include higher-derivative theories with terms like $R^{ab} R_{ab}$, see [16].

We consider a spatially homogeneous and isotropic Friedmann world model with the most general spacetime dependent perturbations

$$ds^2 = -(1 + 2\alpha) dt^2 - 2a (\beta_{,\alpha} + B_\alpha) dt dx^\alpha + a^2 \Big[g^{(3)}_{\alpha\beta} (1 + 2\varphi) + 2\gamma_{,\alpha|\beta} + 2C_{(\alpha|\beta)} + 2C_{\alpha\beta} \Big] dx^\alpha dx^\beta. \quad (2)$$

α, β, γ, and φ indicate the scalar-type structure; the transverse B_α and C_α indicate the vector-type structure; the transverse-tracefree $C_{\alpha\beta}$ indicates the tensor-type structure. The three types of structures are related to the density condensation, the rotation, and the gravitational wave, respectively. Since, to the linear order in the Friedmann background, these three types of structures evolve independently, we can handle them separately. Indices of B_α, C_α and $C_{\alpha\beta}$ are based on $g^{(3)}_{\alpha\beta}$; a vertical bar indicates a covariant derivative based on $g^{(3)}_{\alpha\beta}$.

We also consider the general perturbations in the hydrodynamic energy-momentum tensor and the scalar field $T_{ab}(\mathbf{x},t) = \bar{T}_{ab}(t) + \delta T_{ab}(\mathbf{x},t)$ and $\phi(\mathbf{x},t) = \bar{\phi}(t) + \delta\phi(\mathbf{x},t)$. The perturbed order energy-momentum tensor in terms of the hydrodynamic fluid quantities is

$$\begin{aligned} &T^0_0 = -\left(\bar{\mu} + \delta\mu\right), \quad T^0_\alpha = \left(\mu + p\right) v_\alpha, \\ &T^\alpha_\beta = \left(\bar{p} + \delta p\right) \delta^\alpha_\beta + \pi^\alpha_\beta, \end{aligned} \tag{3}$$

where v_α and π^α_β are based on $g^{(3)}_{\alpha\beta}$.

We introduce the following gauge-invariant combinations

$$\begin{aligned} &\varphi_v \equiv \varphi - \frac{aH}{k} v, \quad \varphi_{\delta\phi} \equiv \varphi - \frac{H}{\dot{\phi}} \delta\phi \equiv -\frac{H}{\dot{\phi}} \delta\phi_\varphi, \\ &\varphi_\chi \equiv \varphi - H\chi, \end{aligned} \tag{4}$$

where k is a comoving wavenumber and $H \equiv \dot{a}/a$; an overdot and a prime indicate time derivatives based on t and the conformal time η, respectively, with $dt \equiv a d\eta$. φ_χ is the same as φ in the zero-shear gauge which sets $\chi \equiv a(\beta + a\dot{\gamma})$ equals to zero as the gauge condition; φ_v is the same as φ in the comoving gauge condition which takes $v/k \equiv 0$ as the gauge condition; v introduced as $v_\alpha \equiv -v_{,\alpha}/k$ is a velocity related scalar-type perturbation variable. For the scalar field the velocity related effective fluid quantity becomes $a(\mu+p)v/k = \dot{\phi}\delta\phi$, thus the uniform-field gauge with $\delta\phi \equiv 0$ coincides with the comoving gauge condition [17]. The gauge-invariant combination φ_v was first introduced by Lukash in 1980 [4]; in the following we will notice the profound importance of φ_v, the Lukash variable, in handling the scalar-type cosmological perturbations.

The equations for background are:

$$H^2 = \frac{1}{3F}\left[\mu + \frac{1}{2}\left(\omega\dot{\phi}^2 - f + RF + 2V\right) - 3H\dot{F}\right] - \frac{K}{a^2}, \tag{5}$$

$$\dot{H} = -\frac{1}{2F}\left(\mu + p + \omega\dot{\phi}^2 + \ddot{F} - H\dot{F}\right) + \frac{K}{a^2}, \tag{6}$$

$$R = 6\left(2H^2 + \dot{H} + \frac{K}{a^2}\right), \tag{7}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{2\omega}\left(\omega_{,\phi}\dot{\phi}^2 - f_{,\phi} + 2V_{,\phi}\right) = 0, \tag{8}$$

$$\dot{\mu} + 3H\left(\mu + p\right) = 0, \tag{9}$$

where $F \equiv \partial f/(\partial R)$. Equation (6) follows from the rest of the equations. K is the sign of the background spatial curvature. In Einstein gravity limit we have $F = 1/(8\pi G)$. Our gravity theory includes the cosmological constant, Λ*.

III. CLASSICAL EVOLUTION

We consider *near flat* background, thus neglect K term. The equations and the large-scale solutions for the scalar- and tensor-type structures can be written in a unified form as

$$\frac{1}{a^3 Q}(a^3 Q\dot{\Phi})^{\cdot} + c_A^2 \frac{k^2}{a^2}\Phi = 0, \tag{10}$$

where, for the fluid in Einstein gravity, the field in generalized gravity, and the tensor-type structures, respectively, we have [18,10,12]:

$$\Phi = \varphi_v, \quad Q = \frac{\mu + p}{c_A^2 H^2}, \qquad c_A^2 = c_s^2 \tag{11}$$

$$\Phi = \varphi_{\delta\phi}, \quad Q = \frac{\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F}}{\left(H + \frac{\dot{F}}{2F}\right)^2} \equiv \frac{\dot{\phi}^2}{H^2} Z_s, \quad c_A^2 = 1, \tag{12}$$

$$\Phi = C^\alpha_\beta, \quad Q = F \equiv \frac{1}{8\pi G} Z_t, \qquad c_A^2 = 1, \tag{13}$$

where Z's become unity in the limit of Einstein gravity†. Equations (11,12) are valid for single component fluid and field, whereas eq. (13) is valid in the presence of arbitrary numbers of fluid and field as long as the tensor-type anisotropic stress vanishes. The case of eq. (11) is valid for an ideal fluid in Einstein gravity with $c_s^2 \equiv \dot{p}/\dot{\mu}$‡. The case of eq. (12) is valid for the second-order gravity system such as either $f = F(\phi)R$ in the presence of a field ϕ or $f = f(R)$ without the field. The case of eq.

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*It can be simulated using either the scalar field or the fluid. Using the scalar field we let $V \to V + \Lambda/(8\pi G)$. Using the fluid we let $\mu \to \mu + \Lambda/(8\pi G)$ and $p \to p - \Lambda/(8\pi G)$. This causes a change only in eq. (5).

†In the gravity with stringy correction terms

$$\xi(\phi)[c_1 R^2_{GB} + c_2 G^{ab}\phi_{;a}\phi_{;b} + c_3\Box\phi\phi^{;a}\phi_{;a} + c_4(\phi^{;a}\phi_{;a})^2], \tag{14}$$

$$g(\phi)R\tilde{R}, \tag{15}$$

in the Lagrangian, where $R^2_{GB} \equiv R^{abcd}R_{abcd} - 4R^{ab}R_{ab} + R^2$ and $R\tilde{R} \equiv \eta^{abcd}R_{ab}{}^{ef}R_{cdef}$, we still have eq. (10) with more complicated Q and c_A^2 [19].

‡In the situation with general K, with [20]

$$\Phi \equiv \varphi_v - \frac{K/a^2}{4\pi G(\mu + p)}\varphi_\chi, \tag{16}$$

eqs. (10,11) are valid for an ideal fluid [18], whereas the same equations with $c_A^2 \equiv 1 - 3(1 - c_s^2)K/k^2$ (where c_s^2 is for the field) are valid for a minimally coupled scalar field [21].

(13) is valid for the general system in eq. (1). Using $z \equiv a\sqrt{Q}$ and $v \equiv z\Phi$ eq. (10) becomes [4,20]

$$v'' + \left(c_A^2 k^2 - z''/z\right) v = 0. \tag{17}$$

In the large-scale limit, with $z''/z \gg c_A^2 k^2$, we have an exact solution

$$\Phi = C(\mathbf{x}) - D(\mathbf{x}) \int_0^t \frac{dt}{a^3 Q}. \tag{18}$$

Ignoring the transient solution we have a temporally conserved behavior

$$\Phi(\mathbf{x}, t) = C(\mathbf{x}). \tag{19}$$

For the scalar-type perturbation we can show that the decaying solution in eq. (18) is $(\frac{k}{aH})^2$ higher order compared with the one in the zero-shear gauge [10]. Therefore, the non-transient solutions of Φ in the large-scale limit is generally *conserved*. These conservation properties are valid considering generally time varying $p(\mu)$, $V(\phi)$, $\omega(\phi)$, and $f(\phi, R)$ [$F(\phi)$ for $\varphi_{\delta\phi}$ and $f(R)$ for $\varphi_{\delta F}$], thus are valid independently of changes in underlying gravity theory. The unified analyses of the gravity theories belonging to eq. (1) are crucially important to make this point: that is, since the solutions and the conservation properties are valid considering general p, V, ω, and f, we can claim that Φ remains conserved independently of changing equation of state, field potential, and gravity sector.

IV. QUANTUM GENERATION

We have shown that the growing solution of Φ is conserved in the large scale limit *independently* of the specifics of the gravity theories including changes between different gravity theories. Thus, the classical evolution in the large scale is characterized by the conserved quantity $C(\mathbf{x})$ which encodes the information about the spatial structure of the nontransient solution. In order to have information about large scale structure, we need the information about $\Phi = C(\mathbf{x})$ which must have been generated from quantum fluctuations in the early inflationary stage of the universe; gravity alone cannot generate the seed fluctuations out of the spatially homogeneous and isotropic background.

We consider the quantum generation process in unified forms. From eq. (10) we can construct the perturbed action in a unified form [4,5,8,11,12]

$$\delta^2 S = \frac{1}{2} \int a^3 Q \left(\dot{\Phi}^2 - c_A^2 \frac{1}{a^2} \Phi^{|\gamma} \Phi_{,\gamma} \right) dt d^3x. \tag{20}$$

This action as well as eqs. (10,17) was first derived by Lukash in 1980 in the context of an ideal fluid [4], and later was derived in the context of a field [5]; eq. (17) first appeared in the work by Field and Shepley in 1968 [20].

In order to handle the quantum mechanical generations of the scalar-type structure and the gravitational wave, we regard the perturbed parts of the metric and matter variables as Hilbert space operators, $\hat{\Phi}(\mathbf{x}, t)$. Since we are considering a flat three-space background, we may expand $\hat{\Phi}$ in mode function expansion

$$\hat{\Phi}(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[\hat{a}_k \Phi_k(t) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_k^\dagger \Phi_k^*(t) e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \tag{21}$$

where $\Phi_k(t)$ is a mode function. The annihilation and creation operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ follow the standard commutation relations. In the quantization process of the gravitational wave we need to take into account of the two polarization states properly [3,12]. From our perturbed action in eq. (20) we have $\pi_\Phi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = a^3 Q \dot{\Phi}$. From the equal-time commutation relation $[\hat{\Phi}(\mathbf{x}, t), \hat{\pi}_\Phi(\mathbf{x}', t)] = i\delta^3(\mathbf{x} - \mathbf{x}')$ we can derive

$$\Phi_k \dot{\Phi}_k^* - \Phi_k^* \dot{\Phi}_k = i/(a^3 Q). \tag{22}$$

Under the *ansatzs*[§]

$$z''/z = n/\eta^2, \quad c_A^2 = \text{constant}, \tag{23}$$

the mode function has an exact solution

$$\Phi_k(\eta) = \frac{\sqrt{\pi|\eta|}}{2a\sqrt{Q}} \left[c_1(k) H_\nu^{(1)}(x) + c_2(k) H_\nu^{(2)}(x) \right], \tag{24}$$

where $\nu \equiv \sqrt{n + 1/4}$ and $x \equiv c_A k |\eta|$. From the quantization condition we have

$$|c_2|^2 - |c_1|^2 = 1, \tag{25}$$

where for the gravitational wave this condition should be met for each polarization state [12]. The power spectrum based on the vacuum expectation value of $\hat{\Phi}$ is

$$\begin{aligned} \mathcal{P}_{\hat{\Phi}}(k, \eta) &\equiv \frac{k^3}{2\pi^2} \int \langle \hat{\Phi}(\mathbf{x} + \mathbf{r}, t) \hat{\Phi}(\mathbf{x}, t) \rangle_{\text{vac}} e^{-i\mathbf{k}\cdot\mathbf{r}} d^3r \\ &= \frac{k^3}{2\pi^2} |\Phi_k(\eta)|^2. \end{aligned} \tag{26}$$

Assuming the simplest vacuum state with $c_2 = 1$ and $c_1 = 0$ which corresponds to the flat spacetime quantum field theory vacuum state with positive frequencies, in the large-scale limit we have[**]

$$\mathcal{P}_{\hat{\Phi}}^{1/2}\Big|_{LS} = \frac{H}{2\pi} \frac{1}{aH|\eta|} \frac{\Gamma(\nu)}{\Gamma(3/2)} \left(\frac{k|\eta|}{2} \right)^{3/2-\nu} \frac{1}{c_A^\nu \sqrt{Q}}, \tag{27}$$

[§]For solutions in the case of more general ansatz, see [22].
[**]For $\nu = 0$ we have an additional $2\ln(c_A k|\eta|)$ factor.

where we should consider additional $\sqrt{2}$ factor for the gravitational wave which follows from proper considering of the two polarization states [12]. We have $\mathcal{P}_{\hat{\Phi}}|_{LS} =$ constant[††], thus consistent with the general large-scale behavior in eq. (19). The spectral indices are

$$n_S - 1 \equiv \frac{d\ln \mathcal{P}_{\varphi_v}}{d\ln k} = 3 - 2\nu_s,$$
$$n_T \equiv \frac{d\ln \mathcal{P}_{C^\alpha_\beta}}{d\ln k} = 3 - 2\nu_t. \quad (28)$$

V. SLOW-ROLL INFLATION

We consider situations without the fluid, thus $c_A^2 = 1$. We introduce the slow-roll parameters [10]

$$\epsilon_1 \equiv \frac{\dot{H}}{H^2}, \quad \epsilon_2 \equiv \frac{\ddot{\phi}}{H\dot{\phi}}, \quad \epsilon_3 \equiv \frac{1}{2}\frac{\dot{F}}{HF},$$
$$\epsilon_4 \equiv \frac{1}{2}\frac{\dot{E}}{HE}, \quad E \equiv F\left(\omega + \frac{3\dot{F}^2}{2\dot{\phi}^2 F}\right). \quad (29)$$

Compared with the Einstein gravity in [23] we have two additional parameters ϵ_3 and ϵ_4 for the scalar-type perturbation which reflect the effects of additional parameters F and ω in our generalized gravity; for the tensor-type perturbation we have one additional parameter ϵ_3 from F. From eqs. (12,13) we have [10,12]

$$\frac{z_s''}{z_s} = a^2\Big[H^2\left(1-\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4\right)\left(2+\epsilon_2-\epsilon_3+\epsilon_4\right)$$
$$+H\left(-\dot{\epsilon}_1+\dot{\epsilon}_2-\dot{\epsilon}_3+\dot{\epsilon}_4\right) - 2(\frac{3}{2}-\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4)$$
$$\times H\frac{\dot{\epsilon}_3}{1+\epsilon_3} - \frac{\ddot{\epsilon}_3}{1+\epsilon_3} + 2\frac{\dot{\epsilon}_3^2}{(1+\epsilon_3)^2}\Big], \quad (30)$$
$$\frac{z_t''}{z_t} = a^2\left[H^2\left(1+\epsilon_3\right)\left(2+\epsilon_1+\epsilon_3\right) + H\dot{\epsilon}_3\right], \quad (31)$$

and $\int^\eta (1+\epsilon_1)d\eta = -1/(aH)$.

Assuming $\dot{\epsilon}_i = 0$ we have $(1+\epsilon_1)\eta = -1/(aH)$, thus ansatzs in eq. (23) are satisfied with

$$n_s = \frac{(1-\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4)(2+\epsilon_2-\epsilon_3+\epsilon_4)}{(1+\epsilon_1)^2},$$
$$n_t = \frac{(1+\epsilon_3)(2+\epsilon_1+\epsilon_3)}{(1+\epsilon_1)^2}. \quad (32)$$

In such a case the rest of the exact results in §IV are available. Since the large-scale structures are generated during short time interval (about 60 e-folds) of the latest inflation, we anticipate time variation of ϵ_i during that period is negligible; still this is an *assumption* we are making in the following. Under this situation the power-spectra of the two-types of structures in the large-scale limit are given in eq. (27) with the spectral indices given as

$$n_S - 1 = 3 - \sqrt{4n_s+1}, \quad n_T = 3 - \sqrt{4n_t+1}. \quad (33)$$

Thus, by imposing the condition of Zel'dovich specra ($n_S - 1 \simeq 0 \simeq n_T$) which is consistent with the CMBR observation, we can derive constraints on ϵ_i's, thus on the parameters of the gravity theory (V, ω, and F).

Now, to the first-order in the slow-roll parameters, i.e., further *assuming* $|\epsilon_i| \ll 1$, from eq. (27) we can derive

$$\mathcal{P}^{1/2}_{\hat{\varphi}_{\delta\phi}}\Big|_{LS} = \frac{H}{|\dot{\phi}|}\mathcal{P}^{1/2}_{\delta\hat{\phi}_\varphi}\Big|_{LS} = \frac{H^2}{2\pi|\dot{\phi}|}\frac{1}{\sqrt{Z_s}}\Big\{1+\epsilon_1$$
$$+\big[\gamma_1 + \ln\left(k|\eta|\right)\big](2\epsilon_1-\epsilon_2+\epsilon_3-\epsilon_4)\Big\}, \quad (34)$$
$$\mathcal{P}^{1/2}_{\hat{C}^\alpha_\beta}\Big|_{LS} = \sqrt{16\pi G}\frac{H}{2\pi}\frac{1}{\sqrt{Z_t}}$$
$$\times\Big\{1+\epsilon_1+\big[\gamma_1+\ln\left(k|\eta|\right)\big](\epsilon_1-\epsilon_3)\Big\}, \quad (35)$$

where $\gamma_1 \equiv \gamma_E + \ln 2 - 2 = -0.7296\ldots$, with γ_E the Euler constant. We have [we have corrected an error in Z_s in the published version]

$$Z_s = \frac{E/F}{(1+\epsilon_3)^2}, \quad Z_t = 8\pi G F. \quad (36)$$

Thus, besides ϵ_1, the scalar-type perturbation is affected by ϵ_2, ϵ_3 and ϵ_4 (thus, ϕ, F and ω), whereas the tensor-type perturbation is affected by ϵ_3 (thus, F) only; see also eq. (38).

The observationally relevant scales exit Hubble horizon within about 60 e-folds before the end of the latest inflation. Far outside the horizon the quantum fluctuations classicalize and we can identify $\mathcal{P}_\Phi = \mathcal{P}_{\hat{\Phi}}$ where $\mathcal{P}_\Phi$ is the power-spectrum based on spatial averaging

$$\mathcal{P}_\Phi(k,\eta) \equiv \frac{k^3}{2\pi^2}\int \langle\Phi(\mathbf{x}+\mathbf{r},t)\Phi(\mathbf{x},t)\rangle_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{r}} d^3r$$
$$= \frac{k^3}{2\pi^2}|\Phi(k,\eta)|^2, \quad (37)$$

with $\Phi(k,\eta)$ a Fourier transform of $\Phi(\mathbf{x},\eta)$. Since Φ is conserved in the large-scale limit, the power-spectra in eqs. (34,35) can be *identified* as the classical power-spectra at later epoch. We have in mind a scenario in which the inflation based on a field or a generalized gravity is followed by ordinary radiation and matter dominated eras based on Einstein gravity. We have shown in eq. (19) that as long as the scale remains in the super-horizon scale Φ is conserved independently of the changing gravity theory from one type to the other. Therefore,

[††] Using $n \equiv q(q+1)$ we can show $z \propto |\eta|^{-q}$ and $\nu = q + \frac{1}{2}$, thus $\mathcal{P}_{\hat{\Phi}}|_{LS} =$ constant for $\nu > 0$; for $\nu = 0$ we additionally have $z \propto \sqrt{|\eta|}\ln|\eta|$, thus $\mathcal{P}_{\hat{\Phi}}|_{LS} =$ constant as well.

eqs. (34,35) are now valid for the classical power-spectra. The spectral indices of the scalar and tensor-type perturbations in eq. (28) become

$$n_S - 1 = 2(2\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4), \quad n_T = 2(\epsilon_1 - \epsilon_3). \quad (38)$$

For the scale independent Zel'dovich ($n_S - 1 \simeq 0 \simeq n_T$) spectra the quadrupole anisotropy becomes

$$\langle a_2^2 \rangle = \langle a_2^2 \rangle_S + \langle a_2^2 \rangle_T = \frac{\pi}{75} \mathcal{P}_{\varphi_{\delta\phi}} + 7.74 \frac{1}{5} \frac{3}{32} \mathcal{P}_{C_{\alpha\beta}}, \quad (39)$$

which is valid for $K = 0 = \Lambda$; for a general situation with nonvanishing Λ, see [24]. The four-year *COBE*-DMR data give $\langle a_2^2 \rangle \simeq 1.1 \times 10^{-10}$, [25]. From eqs. (34,35) the ratio between two types of perturbations $r_2 \equiv \langle a_2^2 \rangle_T / \langle a_2^2 \rangle_S$ becomes [we have corrected an error in eq. (40) and the rest of this paragraph in the published version‡‡]

$$\begin{aligned} r_2 &= 13.8 \times 4\pi G \frac{\dot{\phi}^2}{H^2} \left| \frac{Z_s}{Z_t} \right| \\ &= 13.8 \frac{1}{(1+\epsilon_3)^2} \left| \frac{\omega \dot{\phi}^2}{2H^2 F} + 3\epsilon_3^2 \right| \\ &= \left| -13.8 \frac{1}{(1+\epsilon_3)^2} \left[(\epsilon_1 - \epsilon_3)(1+\epsilon_3) - \frac{\dot{\epsilon}_3}{H} \right] \right| \\ &= \left| -13.8 (\epsilon_1 - \epsilon_3) \right|, \end{aligned} \quad (40)$$

where in the last step we used the linear slow-roll condition. In the limit of Einstein gravity we have $r_2 = -13.8\epsilon_1 = -6.92 n_T$ which is independent of V and is the well known consistency relation; ϵ_1 is always negative in the ordinary slow-roll inflation, thus n_T is negative. Even in our class of generalized gravity theories, from eqs. (38,40) we have $r_2 = |-13.8(\epsilon_1 - \epsilon_3)| = |-6.92 n_T|$, thus the consistency relation (in the amplitude!) remains valid. However, in our generalized gravity case, n_T in eq. (38) could have either sign depending on situations.

Inflation based on Einstein gravity with a minimally coupled scalar field is a simple case with $F = 1/(8\pi G)$ and $\omega = 1$. In this case we have $\epsilon_3 = 0 = \epsilon_4$ and $Z = 1$. The power spectra of slow-roll inflation [23] belong to eqs. (34,35,38). Accuracy of the slow-roll approximation compared with the exact integration of the fundamental equation in eq. (10) has been discussed in [27]. For a recent attempt to consider higher-order effects of the slow-roll parameters in a perturbative approach, thus going beyond the ansatz made in eq. (23), see [28].

The presence of F and ω, thus ϵ_3, ϵ_4 and Z's in eqs. (34,35,38,40) indicates the deviation from the Einstein gravity. Inflationary spectra in various inflationary models based on generalized gravity theories made in [29,26] can be recovered by simply reducing our general results in this paper.

VI. DISCUSSIONS

As we have shown in this paper, even in a class of generalized gravity theories included in eq. (1) we can present the results quite similarly as in Einstein gravity case and in unified forms. The effects of generalized gravity appear in two additional parameters F and ω which are reflected in the two additional slow-roll parameters ϵ_3 and ϵ_4. One important underlying reason for such simple results in apparently complicated and diverse gravity theories belonging to eq. (1) can be traced to the conformal transformation property of the gravity theory we are considering [30].

In addition to the coherent and unified presentation of the classical evolution and quantum generation processes, the slow-roll power spectra in §V can be regarded as new contributions of the present work. These results, in fact, include results from most of the inflationary scenarios based on generalized gravity as well as Einstein gravity theory. In generic forms, eqs. (34,35,38) show the amplitudes and spectral indices of the generated structures, and eq. (40) shows the ratio of gravitational wave contribution relative to the scalar-type structure. Various previous studies on the subject can be regarded as specific limits of these generic results.

ACKNOWLEDGMENTS

We thank Cyril Cartier for useful information and Ewan Stewart for useful comments. We also wish to

‡‡We thank David Wands for pointing out a possible error in eq. (40) of the previous version based on the conformal transformation argument.

Using the conformal transformation properties in eqs. (10-14) of [11] we can show that the slow-roll parameters transform to the linear order as

$$\hat{\epsilon}_1 = \epsilon_1 - \epsilon_3, \quad \hat{\epsilon}_2 = \epsilon_2 - 3\epsilon_3 + \epsilon_4,$$

where hats denote quantities in the conformally transformed Einstein frame. Thus, we have

$$\begin{aligned} &\hat{n}_S - 1 = 2(2\hat{\epsilon}_1 - \hat{\epsilon}_2) = 2(2\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4) = n_S - 1, \\ &\hat{n}_T = 2\hat{\epsilon}_1 = 2(\epsilon_1 - \epsilon_3) = n_T, \\ &\hat{r}_2 = | -13.8\hat{\epsilon}_1 | = | -13.8(\epsilon_1 - \epsilon_3) | = r_2. \end{aligned}$$

These results are consistent with the fact that $\varphi_{\delta\phi}$ and $C_{\alpha\beta}$ are invariant under the conformal transformation as shown in eq. (25) of [11].

In fact, using the conformal transformation properties in eqs. (10-14,25) of [11] we can check that eq. (10) and eq. (20), thus all the consequent results in the classical evolution and the quantum generation, can be simply drived from the known results in the Einstein frame. This does not mean that the results in the two (the original and the Einstein frames) are equivalent. It only means that the results in the two frames are related through the conformal transformation.

thank Dominik Schwarz for useful discussions and insightful comments. JH was supported by the Korea Research Foundation Grants (KRF-2000-013-DA004 and 2000-015-DP0080). HN was supported by grant No. 2000-0-113-001-3 from the Basic Research Program of the Korea Science and Engineering Foundation.

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